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ON 3-CHROMATIC HYPERGRAPHS

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Let $\mathcal{F}$ be n-uniform hypergraph. In the present paper I prove that if $|\mathcal{F}|<n^{1/3-\delta}2^n$ then the chromatic number of $\mathcal{F}$ is equal to 2. I have a result in the general (not necessarily uniform) case too.

1. Introduction

A hypergraph is a collection of sets. This paper deals with finite hypergraphs only. The sets in the hypergraph are called edges, and the elements of these edges are points. The hypergraph is n-uniform if every edge has n points.

The chromatic number of a hypergraph is the least number k, such that the points can be k-colored so that no edge is monochromatic In our understanding any map $f:G\to\{1,\ldots,k\}$ is a k-coloration of a hypergraph, i.e. monochromatic edges are also allowed.

Let $m(n)$ be the minimum number of edges of a 3-chromatic n-uniform hypergraph. It is known [1, 3]

$$\tfrac{1}{5}\log n 2^n \leq m(n) \leq n^2 2^n.$$

We improve the lower bound showing

Theorem 1.1. *For every $\delta>0$ if $n\geq n(\delta)$*

$$m(n)\geq n^{\frac{1}{3}-\delta}2^n.$$

It would be very interesting to improve the upper bound.

In the general (not necessarily uniform) case we have no good estimation. Let $\{E_k\}_{k=1}^N$ be a 3-chromatic hypergraph. Let

$$f(n)=\min\left(\sum_{k=1}^{N}2^{-|E_k|}\right),$$

where the minimum is extended over all hypergraphs with $\min|E_k|=n$. Erdös [2] proved that $f(n) \geq \frac{1}{2}$ and he [2, 4] suggested that $f(n) \to \infty$ as $n \to \infty$. We shall prove this.

Theorem 1.2. $\lim_{n \to +\infty} f(n) = +\infty$.

It is clear that $f(n) \leq m(n) 2^{-n}$. We could not decide whether $f(n) < m(n) 2^{-n}$.

2. Proof of Theorem 1.1

Let $\mathscr{F}$ be n-uniform hypergraph. If $n \geq n(\varepsilon)$ and

$$|\mathscr{F}| < n^{\frac{1}{3}-\delta} 2^n, \tag{2.1}$$

then we show that the points can be 2-colored so that no edge is monochromatic.

For every 2-coloration $C = (K_1, K_2)$ of $\mathscr{F}$ we define a sequence of relations as follows

$$E R^i(C, j) F \quad E, F \in \mathscr{F}; \quad j = 1, \ldots, n; \quad i = 1, 2$$

holds if and only if the coloration C satisfies the conditions

(a) $E \subset K_i$,
(b) $E \cap F \neq \emptyset$ and $|F \cap K_i| = j$.

Let

$$\mathscr{A}^i(C) = \{E \in \mathscr{F} \colon E \subset K_i\},$$

$$\mathscr{B}^i(C, j) = \{(E, F) \colon E R^i(C, j) F\}.$$

Let

$$|a|^+ = \begin{cases} a & \text{if } a > 0, \\ 0 & \text{else} \end{cases}$$

Lemma 2.1. *There is a 2-coloration $C^* = (K_1^*, K_2^*)$ of $\mathscr{F}$ such that*

$$|\mathscr{A}^i(C^*)| \leq 4 n^{\frac{1}{3}-\delta} \quad i = 1, 2$$

and

$$|\mathscr{B}^i(C^*, j)| \leq n^{-\frac{1}{3}} \binom{n}{j} 2^{|j - \log n|^+} \quad i = 1, 2; \ 1 \leq j \leq \sqrt{n}.$$

Proof. Let $\mathscr{C}$ denote the set of all 2-colorations of $\mathscr{F}$ and denote by p the number of points of $\mathscr{F}$ (obviously $|\mathscr{C}| = 2^p$).

We have by (2.1)

$$\sum_{C \in \mathscr{C}} |\mathscr{A}^i(C)| = \sum_{E \in \mathscr{F}} |\{C \in \mathscr{C} \colon E \in \mathscr{A}^i(C)\}|$$

$$= \sum_{E \in \mathscr{F}} 2^{p-n} = |\mathscr{F}| 2^{p-n} < n^{\frac{1}{3}-\delta} 2^p,$$

thus

$$|\{C\in\mathscr{C}:\exists_i(|\mathscr{A}^i(C)|>4n^{\frac{1}{2}-\delta})\}|<2^{p-}\ . \tag{2.2}$$

Let

$$\mathscr{F}^{(l)}=\{(E,F):|E\cap F|=l \text{ and } E,F\in\mathscr{F}\}\quad l=1,\dots,n.$$

We have

$$\sum_{C\in\mathscr{C}}|\mathscr{B}^i(C,j)|=\sum_{l=1}^{j}\sum_{(E,F)\in\mathscr{F}^{(l)}}|\{C\in\mathscr{C}:E\mathscr{R}^i(C,j)F\}|$$
$$=\sum_{l=1}^{j}\sum_{(E,F)\in\mathscr{F}^{(l)}}\binom{n-l}{j-l}2^{p-2n+l}<|\mathscr{F}|^2\sum_{l=1}^{j}\binom{n-l}{j-l}2^{p-2n+l}. \tag{2.3}$$

Since

$$\binom{n-l}{j-l}\leqslant\left(\frac{j}{n}\right)^l\binom{n}{j}\quad\text{and}\quad j\leqslant\sqrt{n}$$

we have

$$\sum_{l=1}^{j}\binom{n-l}{j-l}2^l\leqslant\binom{n}{j}\left(\sum_{l=1}^{j}\left(\frac{2j}{n}\right)^l\right)<\frac{4j}{n}\binom{n}{j}. \tag{2.4}$$

By (2.1), (2.3) and (2.4)

$$\sum_{C\in\mathscr{C}}|\mathscr{B}^i(C,j)|<\frac{4j}{n^{\frac{1}{2}+2\delta}}\binom{n}{j}2^p,$$

therefore:

$$\left|\left\{C\in\mathscr{C}:\exists_i\left(|\mathscr{B}^i(C,j)|>n^{-\frac{1}{2}}\binom{n}{j}2^{|j-\log n|^+}\right)\right\}\right|<2^p\left(\frac{8_j}{n^{2\delta}2^{|j-\log n|^+}}\right). \tag{2.5}$$

From (2.5) we obtain if $n\geqslant n_1(\delta)$

$$\left|\left\{C\in\mathscr{C}:\forall j\in[1,\sqrt{n}]\forall i\in[1,2]\left(|\mathscr{B}^i(C,j)|\right.\right.\right.$$
$$\left.\left.\left.\leqslant n^{-\frac{1}{2}}\binom{n}{j}2^{|j-\log n|^+}\right)\right\}\right|\leqslant 2^p\left(1-\frac{8j}{n^{2\delta}2^{|j-\log n|^+}}\right)<2^{p-1}. \tag{2.6}$$

The lemma immediately follows from (2.2) and (2.6).

For every 2-coloration $C=(K_1,K_2)$ of $\mathscr{F}$ we define

$$M^i(C)=\bigcup_{E\in\mathscr{A}^i(C)}E\quad i=1,2$$

and

$$\mathscr{D}^i(C,j)=\{E\cap M^i(C):E\in\mathscr{F},|E\cap M^i(C)|=j$$

and

$$(E\setminus M^i(C))\cap K_i=\emptyset\}\quad i=1,2;\quad j=1,\dots,n.$$

We claim

$$|\mathscr{D}^i(C,j)| \leqslant |\mathscr{B}^i(C,j)| \quad i=1,2; \quad j=1,\ldots,n. \tag{2.7}$$

Indeed, if $D = F \cap M^i(C) \in \mathscr{D}^i(C,j)$, then there exists $E \in M^i(C)$ for which $F \cap E \neq \emptyset$, therefore

$$(E,F) \in \mathscr{B}^i(C,j). \tag{2.8}$$

By the definition of $\mathscr{D}^i(C,j)$

$$D = F \cap K_i, \tag{2.9}$$

therefore D is determined by F and C. But (2.7) is an immediate consequence of (2.8) and (2.9).

Assume that (2.10) holds.

$$\begin{aligned} &\forall i (i=1,2) \quad \exists T^i \subset M^i(C^*) \\ &\{\forall E \in \mathscr{A}^i(C^*)(T^i \cap E \neq \emptyset) \text{ and} \\ &\forall j (1 \geqslant j \leqslant |T^i|) \forall D \in \mathscr{D}^i(C^*, j)(T^i \not\supset D)\}. \end{aligned} \tag{2.10}$$

Now we can complete the proof of Theorem 1.1. Note that if X, Y are sets $X \triangle Y$ denotes the symmetric difference of X and Y. 2^X denotes the power set of X. Set

$$C^{**} = \{K_1^* \triangle (T^1 \cup T^2), K_2^* \triangle (T^1 \cup T^2)\}.$$

It follows from (10) that in C^{**} there is no monochromatic edge. Therefore it is sufficient to prove (2.10).

Lemma 2. *Let $\mathscr{H}_j$ be j-uniform hypergraphs $j=1,2,\ldots$ and G be an n-uniform hypergraph. Suppose that*

$$\mathscr{H}_j \subset 2^M \quad j=1,2,\ldots \qquad \textit{and} \quad G \subset 2^M.$$

Also suppose

$$|\mathscr{H}_j|\,|G| < \frac{1}{100}\binom{n}{j} 2^{|j-3\log|G|-6|^+}. \tag{$*$}$$

Then there exists $T \subset M$ such that

(i) $|T| = [|M|\, n^{-1}(\log|G|+2)]$ (*integral part*)
(ii) *for every $A \in G: T \cap A \neq \emptyset$*
(iii) *for every j, for every $B \in \mathscr{H}_j : T \not\supset B$.*

Proof. Let $t = [mn^{-1}(\log|G|+2)]$ where $m = |M|$. We have

$$\frac{1}{\binom{m}{t}} |\{U \subset M : |U| = t \text{ and } \exists A \in G\,(U \cap A = \emptyset)\}|$$
$$\leqslant \frac{\binom{m-n}{t}}{\binom{m}{t}} |G| \leqslant \left(1 - \frac{n}{m}\right)^t |G| < e^{-2}. \tag{2.11}$$

On the other hand

$$\frac{1}{\binom{m}{t}}|\{U\subset M: |U|=t \text{ and } \exists B\in\mathcal{H}_j(U\supset B)\}|$$

$$\leq \frac{\binom{m-j}{t-j}}{\binom{m}{t}}|\mathcal{H}_j|=\frac{\binom{t}{j}}{\binom{m}{j}}|\mathcal{H}_j|. \tag{2.12}$$

By (*) we have

$$\sum_{j=1}^{\infty}\frac{\binom{t}{j}}{\binom{m}{j}}|\mathcal{H}_j|<\frac{1}{100|G|}\left\{\sum_{j=1}^{\infty}\frac{\binom{n}{j}\binom{t}{j}}{\binom{m}{j}}2^{|j-3\log|G|-6|^+}\right\}$$

$$\leq\frac{1}{100|G|}\left\{\sum_{j=1}^{[3\log|G|+6]}\frac{(\log|G|+2)^j}{j!}\right\}$$

$$+\frac{1}{100|G|}\left\{\sum_{j>[3\log|G|+6]}\frac{(\log|G|+2)^j}{j!}2^{j-3\log|G|-6|^+}\right\}$$

$$\leq\frac{e^{\log|G|+2}}{100|G|}+\frac{1}{50}<\frac{1}{2}. \tag{2.13}$$

If we combine (2.11), (2.12) and (2.13) this completes the proof of the lemma.

Let $M=M^i(C^*)$, $\quad G=\mathcal{A}^i(C^*)$,

$$\mathcal{H}_j=\begin{cases}\mathcal{D}^i(C^*,j) & \text{if}\quad 1\leq j\leq n^{\frac{1}{3}},\\ \emptyset & \text{otherwise},\qquad i=1,2.\end{cases}$$

We shall check the condition (*). By (2.7) and Lemma 2.1

$$|\mathcal{A}^i(C^*)|\,|\mathcal{D}^i(C^*,j)|\leq 4n^{-\delta}\binom{n}{j}2^{|j-\log n|^+}$$

$$<\frac{1}{100}\binom{n}{j}2^{|j-3\log|\mathcal{A}^i(C^*)|-6|^+}\quad\text{if } n\geq n_2(\delta).$$

We note that

$$\left[\frac{|M^i(C^*)|}{n}(\log|\mathcal{A}^i(C^*)|+2)\right]\leq n^{\frac{1}{3}}. \tag{2.14}$$

Indeed, by Lemma 2.1

$$\frac{|M^i(C^*)|}{n}(\log|\mathcal{A}^i(C^*)|+2)\leq|\mathcal{A}^i(C^*)|\,(\log|\mathcal{A}^i(C^*)|+2)$$

$$\leq n^{\frac{1}{3}}\quad\text{if}\quad n\geq n_3(\delta).$$

By the application of Lemma 2.2 and by (2.14) we obtain that (2.10) holds if $n\geq\max_{1\leq i\leq 3}n_i(\delta)$, hence the proof of Theorem 1.1 is completed.

3. Proof of Theorem 1.2

We start by introducing some notations. let

$$\exp(0) = 1 \quad \text{and} \quad \exp(k) = 2^{\exp(k-1)}.$$

Let

$$\lg n(0) = n \quad \text{and} \quad \lg n(k) = \log\log\{\lg n(k-1)\}$$

where $\log X$ is the binary log of X.

Let $\mathscr{F}$ be a hypergraph for which

$$\min_{E \in \mathscr{F}} |E| = n \tag{3.1}$$

and

$$\sum_{E \in \mathscr{F}} 2^{-|E|} \leqslant L. \tag{3.2}$$

If $n \geqslant \exp(7L+100)$, then we show that the points of $\mathscr{F}$ can be 2-colored so that no edge is monochromatic. We may assume that L is integer.

For every 2-coloration $C = (K_1, K_2)$ of $\mathscr{F}$ we define $\mathscr{A}^i(C)$ and $\mathscr{B}^i(C, s, t)$ $[i = 1, 2;\ s = 1, 2, \ldots;\ t = 1, 2, \ldots]$ as follows

$$\mathscr{A}^i(C) = \{E \in \mathscr{F} : E \subset K_i\},$$

let $\mathscr{B}^i(C, s, t)$ be the set of $(E_0, \ldots, E_s)$ where $\{E_j\}_{j=0}^s$ satisfies the conditions

(a) E_j, $j = 0, \ldots, s$ are different elements of $\mathscr{F}$;
(b) $E_j \subset K_i$, $j = 1, \ldots, s$;
(c) $(E_0 \setminus \bigcup_{j=1}^s E_j) \cap K_i = \emptyset$;
(d) $|\bigcup_{j=0}^s E_j| \geqslant \sum_{j=0}^s |E_j| - \binom{3L}{2} \lg n(t) - 3L$.

Lemma 3.1. *There is a 2-coloration $C^* = (K_1^*, K_2^*)$ of $\mathscr{F}$ such that*
(i) $|\mathscr{A}^i(C^*)| \leqslant 3L$ $\quad i = 1, 2$,
(ii) $|\mathscr{B}^i(C^*, s, t)| \leqslant \beta(s, t)$ $\quad i = 1, 2;\ s = 1, \ldots, 3L;\ t = 1, \ldots, 3L$,
where

$$\beta(s, t) = \beta(s, t, L) = L^{s+3} 2^{\binom{3L}{2} \lg n(t) + 3L + 6}.$$

Proof. Let $\mathscr{C}$ denote the set of all 2-colorations of $\mathscr{F}$ and denote by p the number of points of $\mathscr{F}$ (obviously $|\mathscr{C}| = 2^p$).

We have by (3.2)

$$\sum_{C \in \mathscr{C}} |\mathscr{A}^i(C)| = \sum_{E \in \mathscr{F}} |\{C \in \mathscr{C} : E \in \mathscr{A}^i(C)\}| = \sum_{E \in \mathscr{F}} 2^{p-|E|} \leqslant 2^p L,$$

thus

$$|\{C\in\mathscr{C}:\exists i\in[1,2](|\mathscr{A}^i(C)|>3L)\}|<\tfrac{1}{3}2^p. \tag{3.3}$$

Let $\mathscr{F}^1=\mathscr{F}$ and $\mathscr{F}^k=\mathscr{F}^{k-1}\times\mathscr{F}$. Let

$$\mathscr{F}(s,t,L)=\Big\{(E_0,\ldots,E_s):E_j,\quad j=0,\ldots,s$$

are different elements of $\mathscr{F}$ and

$$\Big|\bigcup_{j=0}^{s}E_j\Big|\geq\sum_{j=0}^{s}|E_j|-\binom{3L}{2}\lg n(t)-3L\Big\}.$$

We have by (3.2)

$$\begin{aligned}\sum_{C\in\mathscr{C}}|\mathscr{B}^i(C,s,t)|&=\sum_{(E_0,\ldots,E_s)\in\mathscr{F}(s,t,L)}|\{C\in\mathscr{C}:(E_0,\ldots,E_s)\in\mathscr{B}^i(C,s,t)\}|\\&=\sum_{(E_0,\ldots,E_s)\in\mathscr{F}(s,t,L)}2^{p-|\bigcup_{j=0}^{s}E_j|}\\&\leq 2^{p+\alpha(L,t)}\Bigg(\sum_{(E_0,\ldots,E_s)\in\mathscr{F}(s,t,L)}2^{-\sum_{j=0}^{s}|E_j|}\Bigg)\\&\leq 2^{p+\alpha(L,t)}\Bigg(\sum_{(E_0,\ldots,E_s)\in\mathscr{F}^{s+1}}2^{-\sum_{j=0}^{s}|E_j|}\Bigg)\\&=2^{p+\alpha(L,t)}\Bigg(\sum_{E\in\mathscr{F}}2^{-|E|}\Bigg)^{s+1}\\&\leq L^{s+1}2^{p+\alpha(L,t)}=2^{p-6}\frac{\beta(s,t)}{L^2},\end{aligned}$$

where

$$\alpha(L,t)=\binom{3L}{2}\lg n(t)+3L.$$

Thus

$$|\{C\in\mathscr{C}:|\mathscr{B}^i(C,s,t)>\beta(s,t)\}|<\frac{2^{p-6}}{L^2},$$

therefore

$$\begin{aligned}&|\{C\in\mathscr{C}:\forall i\in[1,2],\forall s\in[1,3L],\forall t\in[1,3L]:|\mathscr{B}^i(C,s,t)|\leq\beta(s,t)\}|\\&\quad\geq 2^p-\sum_{i=1}^{2}\sum_{s=1}^{3L}\sum_{t=1}^{3L}\frac{2^{p-6}}{L^2}>\frac{2}{3}2^p.\end{aligned} \tag{3.4}$$

(3.3) and (3.4) prove the lemma.

Lemma 3.2. *Let $\mathscr{A}$ be a set-system and $\{\varphi(n)\}_{n=1}^{|\mathscr{A}|-1}$ be a decreasing sequence of real numbers. Then $\mathscr{A}$ can be written in the form:*

$$\mathscr{A}=\{A_{j,k}:1\leq j\leq r,0\leq k<s_j\}\quad s_j\geq 1;j=1,\ldots,r$$

satisfying the following conditions

(i) $|A_{j,0}\cap A_{j,k}|\geq \varphi(\sum_{\nu=1}^{j-1} s_\nu+k)$, $1\leq j\leq r$; $1\leq k<s_j$

(ii) *if* $1\leq j<j_1\leq r$, *then* $|A_{j,0}\cap A_{j_1,k}|<\varphi(\sum_{\nu=1}^{j} s_\nu)$, $0\leq k<s_{j_1}$.

Proof. Let $|\mathscr{A}|=m$. We shall prove the Lemma by induction on m. For $m=1$ it is obvious. Assume $m>1$, and the statement is true for arbitrary $\mathscr{B}$ and $\psi(n)_{n=1}^{|\mathscr{B}|-1}$ with $|\mathscr{B}|<m$.

A very simple argument shows that we can choose different elements $A^{(0)},\ldots,A^{(\omega-1)}$ $(\omega\geq 1)$ of $\mathscr{A}$ satisfying the following requirements:

$$|A^{(0)}\cap A^{(j)}|\geq\varphi(j)\quad 1\leq j<\omega, \tag{3.5}$$

$$|A^{(0)}\cap B|<\varphi(\omega)\quad \text{for every } B\in\mathscr{A}\setminus\{A^{(0)},\ldots,A^{(\omega-1)}\}. \tag{3.6}$$

Let

$$\mathscr{A}^*=\mathscr{A}\setminus\{A^{(0)},\ldots,A^{(\omega-1)}\}\quad\text{and}\quad \varphi^*(n)=\varphi(n+\omega).$$

By the induction hypothesis $\mathscr{A}^*$ can be written in the form

$$\mathscr{A}^*=\{A^*_{j,k}:1\leq j\leq r^*, 0\leq k<s^*_j\},$$

where

$$|A^*_{j,0}\cap A^*_{j,k}|\geq\varphi\Big(\omega+\sum_{\nu=1}^{j-1}s^*_\nu+k\Big), \tag{3.7}$$

$$|A^*_{j,0}\cap A^*_{j_1,k}|<\varphi\Big(\omega+\sum_{\nu=1}^{j}s^*_\nu\Big)\quad\text{if}\quad j<j_1. \tag{3.8}$$

Let

$$A_{1,k}=A^{(k)}\quad 0\leq k<\omega=s_1$$

and

$$A_{j,k}=A^*_{j-1,k}\quad 2\leq j\leq r^*+1,\ 0\leq k<s^*_{j-1}=s_j.$$

Lemma 3.2 follows from (3.5), (3.6), (3.7) and (3.8).

Let

$$\mathscr{A}=\mathscr{A}^i(C^*)\quad i=1,2\qquad\text{and}\quad \varphi(t)=\lg n(t),$$

then by Lemma 3.2 and by (3.1) $\mathscr{A}^i(C^*)$ can be written in the form

$$\mathscr{A}^i(C^*)=\{A^i_{j,k}:1\leq j\leq r^{(i)}, 0\leq k<s^{(i)}_j\}\quad i=1,2$$

where

$$|A^i_{j,0}\cap A^i_{j,k}|\geq\lg n\Big(\sum_{\nu=1}^{j-1}s^{(i)}_\nu+k\Big)\quad 1\leq j\leq r^{(i)}, 0\leq k<s^{(i)}_j \tag{3.9}$$

and

$$|A^i_{j,0}\cap A^i_{j_1,k}|<\lg n\Big(\sum_{\nu=1}^{j} s^{(i)}_\nu\Big)\quad 1\leq j<j_1\leq r^{(i)},\ 0\leq k<s^{(i)}_{j_1}. \tag{3.10}$$

Let

$$\mathcal{A}^i_l=\{A^i_{j,0}: j\geq l\},$$

$$M^i_l=\bigcup_{A\in\mathcal{A}^i_l} A,$$

$$\mathcal{D}^i_l=\mathcal{D}^i_l(C^*)=\{E\in\mathcal{F}:(E\setminus M^i_l)\cap K^*_i=\emptyset,$$
$$|E\cap M^i_1|\leq 3L \text{ and } E\cap M^i_1=E\cap M^i_l\}$$
$$i=1,2;\ l=1,\ldots,r^{(i)}.$$

Lemma 3.3

$$3L\,|D^i_l|<\lg n\Big\{\Big(\sum_{j=1}^{l} s^{(i)}_j\Big)-1\Big\}\quad i=1,2;\ l=1,\ldots,r^{(i)}.$$

Proof. Let $E\in\mathcal{D}^i_l$ and $A^i_{j_1,0}\cdots A^i_{j_q,0}$ $(l\leq j_1<\cdots<j_q)$ denote those elements of $\mathcal{A}^i_l$ which intersect E. By the definition of $\mathcal{D}^i_l$

$$\Big(E\setminus\bigcup_{\nu=1}^{q} A^i_{j_\nu,0}\Big)\cap K^*_i=\emptyset. \tag{3.11}$$

On the other hand by (3.10)

$$\Big|\bigcup_{\nu=1}^{q} A^i_{j_\nu,0}\cup E\Big|\geq|E|+\sum_{\nu=1}^{q}|A^i_{j_\nu,0}|$$
$$-\Big|E\cap\bigcup_{\nu=1}^{q}A^i_{j_\nu,0}\Big|-\sum_{1\leq\nu<\mu\leq q}|A^i_{j_\nu,0}\cap A^i_{j_\mu,0}|$$
$$\geq|E|+\sum_{\nu=1}^{q}|A^i_{j_\nu,0}|-3L-\binom{3L}{2}\lg n\Big(\sum_{j=1}^{l}s^{(i)}_j\Big), \tag{3.12}$$

therefore by (3.11), (3.12)

$$(E, A^i_{j_1,0},\ldots,A^i_{j_q,0})\in\mathcal{B}^i\Big(C^*,q,\sum_{j=1}^{l}s^{(i)}_j\Big). \tag{3.13}$$

Lemma 3.1, the definition of $\lg n(t)$ and (3.13) imply

$$|\mathcal{D}^i_l|\leq\sum_{q=1}^{3L}\Big|\mathcal{B}^i\Big(C^*,q,\sum_{j=1}^{l}s^{(i)}_j\Big)\Big|$$
$$\leq\sum_{q=1}^{3L}\beta\Big(q,\sum_{j=1}^{l}s^{(i)}_j\Big)=2^{\binom{3L}{2}\lg n(\sum_j^l s^{(i)}_j)+3L+6}\Big(\sum_{q=1}^{3L}L^{q+3}\Big)$$
$$<\frac{\lg n\Big\{\Big(\sum_{j=1}^{l}s^{(i)}_j\Big)-1\Big\}}{3L}.$$

The least inequality is true, since by $n \geq \exp(7L+100)$

$$\lg n\left\{\left(\sum_{j=1}^{i} s_j^{(i)}\right)-1\right\} \geq \lg n(3L) \geq \exp(L+100).$$

This completes the proof of the lemma.

Let

$$B_{j,k}^i = (A_{j,0}^i \cap A_{j,k}^i) \setminus \bigcup_{E \in \mathscr{D}_j^i} E.$$

We claim

$$B_{j,k}^i \neq \emptyset \quad i=1,2; \quad 1 \leq j \leq r^{(i)}; \quad 0 \leq k < s_j^{(i)}.$$

Indeed, by (3.9) and Lemma 3.3

$$|B_{j,k}^i| \geq \lg n\left(\sum_{\nu=1}^{j-1} s_\nu^{(i)} + k\right) - 3L\,|\mathscr{D}|$$

$$> \lg n\left(\sum_{\nu=1}^{j-1} s_\nu^{(i)} + k\right) - \lg n\left\{\left(\sum_{\nu=1}^{j} s_\nu^{(i)}\right) - 1\right\} \geq 0.$$

Now we are ready to prove the theorem. Let

$$\tilde{\mathscr{B}} = \{B_{j,k}^i : i=1,2;\ 1 \leq j \leq r^{(i)};\ 0 \leq k < s_j^{(i)}\}$$

and $\tilde{B} = \bigcup_{F \in \tilde{\mathscr{B}}} F$. Let $\tilde{g} : \tilde{\mathscr{B}} \to \tilde{B}$ be a "choice" function so that $\tilde{g}(F) \in F$ for all $F \in \tilde{\mathscr{B}}$ and let $\tilde{g}(\tilde{\mathscr{B}}) = \bigcup_{F \in \tilde{\mathscr{B}}} \tilde{g}(F)$. Set

$$C^{**} = (K_1^* \triangle \tilde{g}(\tilde{\mathscr{B}}),\ K_2^* \triangle \tilde{g}(\tilde{\mathscr{B}}))$$

($X \triangle Y$ denotes the symmetric difference of X and Y).

We claim that in the 2-coloration C^{**} there is no monochromatic edge of $\mathscr{F}$. Observe that it is sufficient to prove that there is no monochromatic edge of $\mathscr{D}_1^i$ $(i=1,2)$.

Let us consider that

$$E_0 \in \mathscr{D}_{j_0}^i \setminus \mathscr{D}_{j_0+1}^i \tag{3.14}$$

is monochromatic in C^{**} (let $\mathscr{D}_j^i = \emptyset$, $j > r^{(i)}$). This means that $E_0 \cap K_i^{**} = \emptyset$ and

$$\exists\, d(d \in (E_0 \cap M_{j_0}^i) \setminus M_{j_0+1}^i). \tag{3.15}$$

Since $d \in K_i^* \setminus K_i^{**}$, we have $d \in \tilde{g}(\tilde{\mathscr{B}})$, so

$$\exists B_{l^*,k}^i \in \tilde{\mathscr{B}} : d = \tilde{g}(B_{l^*,k}^i).$$

If $l^* \leq j_0$, then

$$d \in B_{l^*,k}^i = (A_{l^*,0}^i \cap A_{l^*,k}^i) \setminus \bigcup_{F \in \mathscr{D}_{l^*}^i} F$$

but by (3.14)

$$d \in E_3 \subset \bigcup_{F \in \mathscr{F}^i_{l^*}} F,$$

a contradiction.

Finally, if $l^* \geq j_i + 1$, then

$$d \in B^i_{l^*,k} \subset A^i_{l^*,0} \subset M^i_{j_i+1}$$

but by (3.15)

$$d \notin M^i_{j_i+1},$$

a contradiction.

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